APPROXIMATE CALCULATION OF FREE CONVECTION HEAT TRANSFER IN A RECTANGULAR REGION

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A description is given of an approximate method of calculating heat flux under steady free thermal convection in a rectangular region. This method is based on the assignment of flux and temperature functions in the form of polynomials satisfying the boundary conditions for any values of the coefficients.

The system of equations describing free thermal convection [1, 2],

$$\dot{\bar{v}} + (\bar{v}\nabla)\bar{v} = -\nabla p/\rho + \bar{q}c\theta + \nu\Delta\bar{v},$$

$$\dot{\vartheta} + \bar{v}\nabla\vartheta = \varkappa\Delta\vartheta,$$

$$\dot{\rho} + \nabla(\rho\bar{v}) = 0,$$

may be reduced, for a steady axisymmetric problem with acceleration of body forces varying over the radius, to three dimensionless partial differential equations [3].

It should be noted that the first of the original equations is valid, in the form in which it has been written, only when with uniform temperature distribution the body (gravitational) forces do not give an appreciable pressure gradient. This restriction is removed if the penultimate term of the equation examined is represented in the form $\bar{g}(1 - \beta\vartheta)$.

In the case, simpler than in [3], of a plane problem with constant acceleration of body forces, the original system, as may easily be seen, reduces, for steady motion under conditions of parametric linearization, to two equations,

$$\Delta^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} = G\frac{\partial t}{\partial y}, \quad (1)$$

$$\Delta t + P\left(\frac{\partial\psi}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial t}{\partial x}\right) = 0. \quad (2)$$

The procedure, employed in solving linear equations, of first finding functions which satisfy the equations identically and then taking them in linear combination to satisfy the boundary conditions, is not valid for nonlinear equations such as (1) and (2). To obtain an approximate solution of the system being examined, having assigned a form for each of the unknown functions in a manner satisfying the boundary conditions, we may choose the unknown coefficients so as to satisfy the equations of the system in the best manner.

For thermal convection in a closed region, the boundary conditions for the stream function have the form (the first two equations reflect the absence of motion of the medium—"zero-slip" at the boundary)

$$\frac{\partial\psi}{\partial x} = -V_y = 0, \quad \frac{\partial\psi}{\partial y} = V_x = 0,$$

and also

$$\left.\frac{\partial^2\psi}{\partial x^2}\right|_{y=\pm h} = \left.\frac{-\partial V_y}{\partial x}\right|_{y=\pm h} = 0,$$

$$\left.\frac{\partial^2\psi}{\partial y^2}\right|_{x=\pm 1} = \left.\frac{\partial V_x}{\partial y}\right|_{x=\pm 1} = 0.$$

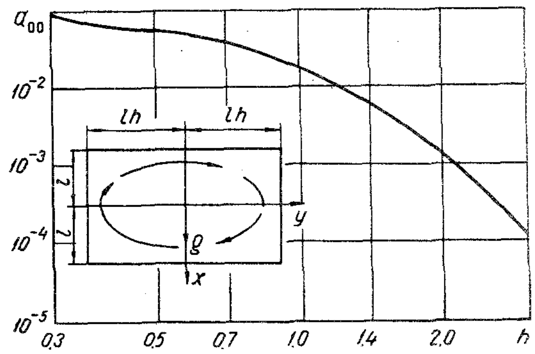

Fig. 1. Schematic of the calculation region and dependence of the scale of convective motion a_{00} on the relative width h of the region when G = 1.

Here the boundaries of the rectangular region are assumed to be parallel to the coordinate axes of Fig. 1. The stream function at the boundary may have an arbitrary constant value, in particular, $\psi = 0$.

The above boundary conditions are satisfied by any polynomial of the form

$$\psi = (x^2 - 1)^2(y^2 - h^2)^2 \sum a_{mn} x^m y^n. \quad (3)$$

The motion described by such a polynomial, for a sufficiently great extent of the sheet in the direction perpendicular to the plane xy, has the form of a cylinder. A similar form of convective motion was examined in [4].

Using this expression, and integrating the left and right sides of Eq. (1) with respect to y over the interval corresponding to the region examined, we obtain

$$16\sum_{n=2k} a_{mn} h^{n+1}\left\{\frac{h^4}{(n+1)(n+3)(n+5)}[m(m-1) - (m-2)(m-3)x^{m-4} - 2(m+2)(m+1)m(m-1)x^{m-2} + (m+4)(m+3)(m+2)(m+1)x^m] + 3(n+1)x^m(x^2-1)^2\right\} +$$

$$+32\sum_{n+q=2k+1} a_{mn}a_{pq}h^{n+q+6}x^{m+p}\Big\{N(n,q)[mx^{-1}-$$

$$-4(m+1)x+6(m+2)x^3-4(m+3)x^5+(m+4)x^7]+$$

$$+\frac{12(n-q)h^2M(m,p,x)}{(n+q)(n+q+2)(n+q+4)(n+q+6)(n+q+8)}\Big\}-$$

$$-G(t|_{y=h}-t|_{y=-h})=0. \qquad (4)$$

Here the first summation is carried out only for even n, and the second only for odd (n + q),

$$M(m,p,x)=$$

$$=m(m-1)[(p-m+2)x^{-3}-2(p-m+4)x^{-1}+$$

$$+(p-m+6)x]-2(m+2)(m+1)[(p-m)x^{-1}-$$

$$-2(p-m+2)x+$$

$$+(p-m+4)x^3]+(m+4)(m+3)[(p-m-2)x-$$

$$-2(p-m)x^3+(p-m+2)x^5],$$

$$N(n,q)=\frac{(n+4)(n+3)(n+2)}{(n+q+6)(n+q+4)(n+q+2)}-$$

$$-\frac{2(n+2)(n+1)n}{(n+q+4)(n+q+2)(n+q)}+$$

$$+\frac{n(n-1)(n-2)}{(n+q+2)(n+q)(n+q-2)}.$$

In conformity with (4), the convective motion is determined only by the temperature difference in the vertical walls (y = ±h). As far as the temperature of the horizontal walls is concerned, they have their own indirect influence on the convection: the larger the difference between the temperatures of the horizontal walls, the larger the temperature difference initially in the ascending and descending streams, and the larger the temperature difference (other conditions being equal) of the corresponding vertical walls.

If the temperature difference is identical over both vertical boundaries of the region, and $t|_{y=h}-t|_{y=-h}\equiv 0$, we may conclude, on the basis of (4), that convective motion is absent. In the cases of practical interest, an ascending stream passes along the heated wall, and its temperature proves to be higher than it is on the wall from which heat is drawn away. With adiabatic vertical walls (having perfect thermal insulation), the temperature of the wall along which the ascending stream passes will also be higher.

Because it is, in general, impossible for any finite values of m and n to satisfy (4) identically, it is necessary, in order to obtain the maximum attainable accuracy, to determine the coefficients in such a way that the value of the integral $\int_{-1}^{1} A^2dx$ is a minimum.

This condition will be satisfied by values of a_{mn} which are roots of the system of equations of the type

$$\frac{\partial}{\partial a_{mn}}\int_{-1}^{1} A^2\,dx=0. \qquad (5)$$

The boundary conditions with respect to temperature are determined by the thermal state of the walls, for example, their temperature. It is then convenient to assign the temperature distribution in the form of a sum of a function t_0 satisfying the boundary temperature conditions, and of a polynomial that vanishes at the boundary,

$$t=t_0+(x^2-1)(y^2-h^2)\sum b_{mn}x^my^n. \qquad (6)$$

In the motionless layers adjacent to the boundaries of the region, the temperature must change along the normal to the boundary faster than in the central zone, where convective motion plays an appreciable role in heat transfer. Therefore a function describing the temperature distribution inside the region being examined must contain terms at least of third degree for each of the variables. It may be expected that in the majority of cases this degree will be sufficient to approximate to the assigned temperature distribution at the boundaries (ten reference points).

In the solution of many problems we require to determine only the heat flux at the boundaries of the region, where the velocity of convective motion is clearly zero. In cases when it is not necessary to obtain accurate data on local velocities of convective motion inside the region, we may confine ourselves to an evaluation of its over-all effect in the heat transfer process. Then, reckoning a direct progressive dependence of the burden of the calculations on the number of terms under the summation sign in (3), it is expedient to seek ψ in the simplest form,

$$\psi=a_{00}(x^2-1)^2(y^2-h^2)^2. \qquad (7)$$

With this form given for ψ, we shall examine the problem for a constant temperature difference of the vertical boundaries $t|_{y=h}-t|_{y=-h}=1$ (e.g., $t_0 = y/2h$). To this end we substitute the left part of (4) into (5), retaining only terms with m = n = 0. It should be noted that any more complex form of assignment of ψ leads to a system of cubic equations for determining the coefficients a_{mn}, while the second sum in (4), in which only terms with odd values of (n + q) appear, vanishes for such a simplified form of ψ, and the coefficient a_{00} is found from the linear equation

$$a_{00}=35\,G\,(1+h^4)/128\,(10+14h^4+7h^8)\,h. \qquad (8)$$

Figure 1 shows graphically the relation between a_{00} and h with G = 1. Using this graph it is easy to find a_{00} also for other values of G, taking into account the linear relation (8) between these quantities.

If we substitute into (2) ψ from (7) and t from (6) with the conditions in question, it takes the form

$$b_{00}[(x^2-1)+(y^2-h^2)]+b_{01}y[3(x^2-1)+$$

$$+(y^2-h^2)]+b_{10}x[(x^2-1)+$$

$$+3(y^2-h^2)]+3b_{11}xy[(x^2-1)+$$

$$+(y^2-h^2)]+2\mathrm{P}a_{00}(x^2-1)(y^2-h^2)\times$$

$$\times\Big\{\frac{x}{2h}(y^2-h^2)+(x^2-1)(y^2-h^2)[b_{10}x(y^2-h^2)$$

$$-b_{10}y(x^2-1)+$$
$$\left.+b_{11}(y^2-h^2)-b_{11}h^2(x^2-1))\right\}=0. \quad (9)$$

This equation will be most accurately satisfied by values of b_{mn} which are roots of the system of equations of type

$$\frac{\partial I}{\partial b_{mn}}=0. \quad (10)$$

From $\partial I/\partial b_{00}=\partial I/\partial b_{11}=0$ we find

$$735b_{00}(3+5h^2+3h^4)-64Pa_{00}b_{11}h^6(1-h^6)=0, \quad (10a)$$
$$891891b_{11}h^2(5+7h^2+5h^4)+327680\,P^2a_{00}^2b_{11}h^{12}-$$
$$-302016\,Pa_{00}b_{00}h^6(1-h^6)=0. \quad (10b)$$

A system of linear homogeneous algebraic equations such as (10a) and (10b) is satisfied by the zeroth-order solution for any ratios of the dimensions of the region and the physical properties of the medium. Therefore, $b_{00}=b_{11}=0$.

From $\partial I/\partial b_{01}=\partial I/\partial b_{10}=0$ we find

$$b_{01}(21+7h^2+h^4)+0.661318\,P^2a_{00}^2b_{01}h^{10}+$$
$$+0.492424\,P^2a_{00}^2h^7-$$
$$-1.82857\,Pa_{00}b_{10}h^4(1-h^2)=0, \quad (10c)$$
$$b_{10}(1+7h^2+21h^4)+0.661318\,P^2a_{00}^2b_{10}h^{10}+$$
$$+0.266667Pa_{00}h^3(2+9h^2)-$$
$$-1.82857Pa_{00}b_{01}h^6(1-h^2)=0. \quad (10d)$$

Knowing P, h, and a_{00}, it is not difficult to find b_{01}, b_{10} from the system (10c) and (10d). A particularly simple solution is obtained for a square region (h = 1), in which the last terms in both these equations become zero, and the system breaks down into two separate equations.

Once the coefficients b_{01} and b_{10} have been found, the calculation may be refined by substituting t from (6) into (1) and repeating the determination of a_{00}, and then of b_{01} and b_{10}.

The local coefficient of heat transfer from the wall to the medium, referred to the difference between the wall temperature and the mean temperature of the medium, is determined from the formula

$$\alpha=-\lambda\frac{\partial t}{\partial n}\Big/t_0.$$

At the vertical boundaries of the region

$$y=\pm h,\ n=\mp ly,\ t_0=\pm\frac{1}{2},$$
$$-\frac{\partial t}{\partial n}=\frac{1}{l}\left[\pm\frac{1}{2h}+2h(x^2-1)(\pm b_{01}h+b_{10}x)\right],$$
$$\alpha=\frac{\lambda}{lh}[1+4h^2(x^2-1)(b_{01}h\pm b_{10}x)].$$

At the horizontal boundaries

$$x=\pm 1,\ n=\mp lx,\quad t_0=\frac{y}{2h},$$
$$-\frac{\partial t}{\partial n}=\frac{2}{l}(y^2-h^2)(b_{01}y\pm b_{10}),$$

$$\alpha=\frac{4\lambda h}{l}(y^2-h^2)\left(b_{01}\pm\frac{b_{10}}{y}\right).$$

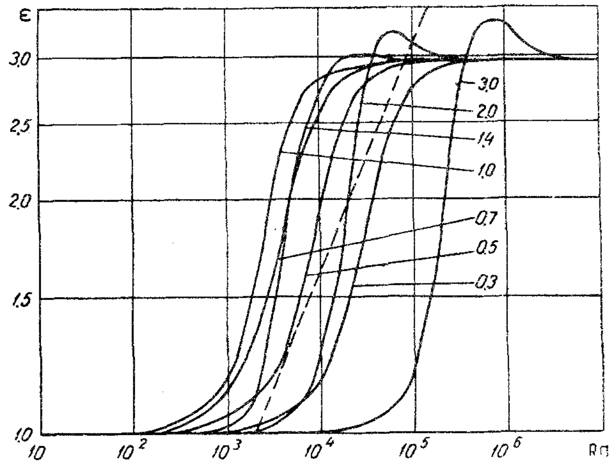

Fig. 2. Dependence of convection coefficient ε on Ra number for a rectangular region. The numerals on the curves correspond to the relative width h of the region. The broken line is the analogous experimental relation for a layer of constant thickness according to [5].

The actual temperature of the medium proceeding toward the horizontal boundaries of the region may differ in isolated sections from the mean temperature level by more than the boundary temperature does at the same point. Therefore, as the foregoing expression also shows, infinitely large and negative values of local heat transfer coefficient are formally possible.

The mean heat flux from unit surface of the vertical boundary is

$$q=-\frac{1}{2}\int_{-1}^{1}\lambda\Theta\left.\frac{\partial t}{\partial n}\right|_{y=h}dx=\frac{\lambda\Theta}{2lh}\left(1-\frac{8}{3}h^3b_{01}\right).$$

In the absence of convection,

$$q=\lambda\Theta/2lh.$$

Thus, the convection coefficient is

$$\varepsilon=1-\frac{8}{3}h^3b_{01}.$$

A comparison is made in Fig. 2 of the curves, constructed using the method of calculation described, of the dependence of ε on Rayleigh number, of which b_{01} is a function, for a rectangular region, and the analogous curve based on the experimental data of [5] for a constant thickness layer. For convenience of the comparison, the reference dimension for both the rectangular region and for the layer has been taken as the distance 2lh between the vertical boundaries.

Comparison of the results for the rectangular region and the layer is valid, because internal circulation loops are created on opposite sides in a plane vertical zone with constant temperature on each wall, due to mutual disturbance of the flow of layers of the medium. There is contrary motion at the junctions of these loops, and therefore the velocity of the medium vanishes at the loop boundary, in a similar manner to what takes place at the horizontal boundaries of the rectangular region.

As was to be expected, the simplified form assumed for assigning ψ does not give a sufficiently accurate description of the process over a wide range of Ra numbers. The variation of convection coefficient as a function of Ra number is suitable for the rectangular region and the layer only for comparatively small values of Ra, up to $\varepsilon \approx 3$.

By using a more complex form of assigning ψ and t, and also by taking certain other steps, the range of usefulness of the above method may be widened, but the results already obtained are evidence that convective transfer in a region increases at the lowest values of Ra, when its form is close to square. It should be noted that the graph for the layer passes between the straight lines corresponding to h = 0.3 and h = 0.5. This is in good agreement with existing data on the longitudinal section of an internal circulation loop (cell) in a gap of length 2 or 3 times its width. Moreover, the fact that the degree of variation of ε as a function of Ra increases with increase of h is interesting. The graphs of Fig. 2 also permit us to obtain more accurate material for preliminary estimation of ε at high Ra values by extrapolation, by use of the analogy between a rectangular region and a slot.

NOTATION

$\bar{v}$—velocity vector of medium; ρ—medium density; p—pressure; $\bar{g}$—acceleration of body forces; ϑ—temperature, computed from some mean level for the case examined; c—temperature coefficient of density; $\nu, \varkappa, \beta$—kinematic viscosity, thermal diffusivity, and volume thermal expansion; x, y—coordinates referred to a definite dimension l, assumed in the given case to be half the extent of the region in the direction of the x axis, which coincides with the direction of action of the body forces; ψ—dimensionless stream function; $t = \vartheta/\Theta$, where Θ is a characteristic temperature difference; $G = g\beta l^3\Theta/\nu^2$ – Grashof number; P—Prandtl number; V_x and V_y—projections on the corresponding coordinate axes of the dimensionless velocity (local Reynolds number); h—ratio of horizontal and vertical dimensions of the region; A—left side of Eq. (4); t_0—a function satisfying the thermal conditions at the boundary; I—integral over the region being examined of the square of the left side of Eq. (9); α—local coefficient of heat transfer from wall to medium, referred to the difference between wall temperature and mean temperature of medium; λ—thermal conductivity of medium; q—heat flux per unit surface; ε—convection coefficient; $Ra = 8h^3GP$—Rayleigh number.

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